

3.1

Chapter 3. Physical Clocks (p k time-of-Arrival

be measured to any accuracy. On the other hand, a direct measurement

same classical time-
measurements

3.2 Free Clocks

We will now attempt to make a measurement of the time-of-arrival. In order to do so, we will need a clock. An ideal clock is linear in time. I.e., the position of the clock's pointer should be proportional to the time t . It is not hard to see that an ideal clock can be represented by the Hamiltonian

$$\mathbf{H}_{clock} = \mathbf{P}_y . \quad (3.40)$$

To read the time of the clock, we measure the coordinate \mathbf{y} conjugate to \mathbf{P}_y . Using the Heisenberg equations of motion we see that the variable \mathbf{y} reads the correct parameter time t found in the Schrödinger equation.

$$\begin{aligned} \mathbf{y}(t) - \mathbf{y}(t_0) &= -i \int [\mathbf{y}, \mathbf{H}_{clock}] dt \\ &= t - t_0 \end{aligned} \quad (3.41)$$

The Hamiltonian for this clock is unbounded from above and below, nonetheless, using a sufficiently massive particle, we can approximate the ideal situation to arbitrary accuracy¹. We write $\mathbf{p} = \langle \mathbf{p} \rangle + \delta$

From (3.41) we see that in order to use this clock to read the time, we need to know the initial position of the clock's dial $\mathbf{y}(t_0)$ and then subtract this from our final reading of \mathbf{y} . Quantum mechanics puts no limitation on how accurately this clock can be measured. If we want to accurately infer the time from the final reading of the clock then the clock must initially be prepared in a state with a very small uncertainty in y . At some later point, we can measure the coordinate $y(t_f)$ to any degree of accuracy we wish to infer the time from $y(t_f) - y(t_0)$. If initially dy is very small, then we know that the time is given by the final reading of y . However, if initially the state of the clock had a large spread in y , then the time we finally obtain will be inaccurate by an amount dy . This means that for this clock, the inaccuracy in the time measurement is given by

$$\delta T = dy \tag{3.43}$$

If we simply want to use this clock to read the time, then there are no restrictions on how accurate the clock can be. So far, nothing prevents us from making the initial state of the clock's pointer as close to an eigenstate of $y(t_0)$ as we desire. However, since $\mathbf{y}(t_0)$ and \mathbf{H}_{clock} do not commute (and cannot commute if the clock is to operate properly), the smaller the uncertainty in $y(t_0)$, the greater the uncertainty in H_{clock} . We will see that if we want to use this clock to measure the time of an event, then we will encounter the limitation given by (3.39). We will need to ensure that initially the position of the clock is uncertain in order for our measurements of the time of an event to succeed.

The reason for this is that since \mathbf{y} is conjugate to $\mathbf{H}_{clock} = \mathbf{P}_y$, accurate clocks (which are narrow in y) have a large spread in P_y . This means that in general the energy of an accurate clock can take on fairly large values. For an infinitely accurate clock the energy will almost always be infinite. Accurate clocks therefore, have a large energy uncertainty, and this makes them very hard to use to measure the time of an event. This is because accurate clocks are usually so energetic that they need a large amount

of energy to turn them off. To measure the time-of-arrival of a particle, the particle itself will have to turn off the clock when it arrives – the external observer cannot supply any energy since she does not know when to turn the clock off. If the clock is much more energetic than the particle, then it will be impossible for the particle to turn off

3.3.1 Measurement with a clock

The simplest model which describes a direct interaction of a particle and a clock [16], without additional “detector” degrees of freedom, is described by the Hamiltonian

$$H = \frac{1}{2m} \mathbf{P}_x^2 + \theta(-x) \mathbf{P}_y. \quad (3.44)$$

Here, the particle’s motion is confined to one spatial dimension, x , and $\theta(x)$ is a step function. The clock’s Hamiltonian is represented by \mathbf{P}_y , and the time is recorded on the

On the other hand, in quantum mechanics the uncertainty relation dictates a strong back-reaction, i.e. in the limit of $\Delta y = \Delta t_A \rightarrow 0$, p_y in (3.45) must have a large uncertainty, and the state of the particle must be strongly affected by the act of measuring.

$\frac{k^2 t}{2m} + pt$. Continuity of ϕ_{kp} requires that

$$\begin{aligned} A_L &= \frac{2k}{k+q} \\ A_R &= \frac{k-q}{k+q}, \end{aligned} \quad (3.51)$$

here $q = \sqrt{k^2 + 2mp} = \sqrt{2m(E_k + p)}$.

The solution of the Schrödinger equation is

$$\psi(x, y, t) = N \int_{-\infty}^{\infty} dk \int_0^{\infty} dp f(p) g(k) \phi_{kp}(x, y, t), \quad (3.52)$$

here N is a normalization constant and $f(p)$ and $g(k)$ are some distributions. For example, with

$$\begin{aligned} f(p) &= e^{-\Delta_y^2 (p-p_0)^2} \\ g(k) &= e^{-\Delta_x^2 (k-k_0)^2 + ikx_0}. \end{aligned} \quad (3.53)$$

and $x_0 > 0$, the particle is initially localized on the left ($x < 0$) and the clock (with probability close to 1) runs. The normalization in eq. (3.52) is thus $N^2 = \frac{\Delta_x \Delta_y}{2\pi^3}$. By choosing $p_0 \approx 1/\Delta_y$, we can now set the clock's energy in the range $0 < p < 2/\Delta_y$.

Let us first show that in the stationary point approximation the clock's final wave function is indeed centered around the classical time-of-arrival. Thus we assume that Δ_y and Δ_x are large such that $f(p)$ and $g(k)$ are sufficiently peaked. For $x > 0$, the integrand in (3.52) has an imaginary phase

$$\theta = qx + kx_0 + py - \frac{k^2 t}{2m} - pt. \quad (3.54)$$

$\frac{d\theta}{dk} = 0$ implies

$$x_{peak}(p) = -\frac{q(k_0)}{k_0} x_0 + \frac{q(k_0)t}{m}, \quad (3.55)$$

and $\frac{d\theta}{dp} = 0$ gives

$$y_{peak}(k) = t - \frac{mx}{q_0}. \quad (3.56)$$

Hence at $x = x_{peak}$ the clock coordinate y is peaked at the classical time-of-arrival

$$y = \frac{mx_o}{k_o}. \quad (3.57)$$

To see that the clock yields a reasonable record of the time-of-arrival, let us consider further the probability distribution of the clock

$$\rho(y, y)_{x>0} = \int dx |\psi(x > 0, y, t)|^2. \quad (3.58)$$

In the case of inaccurate measurements with a small back-reaction on the particle $A_I \simeq 1$. The clocks density matrix is then found (see Appendix B) to be given by:

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \quad (3.59)$$

here the width is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$. As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial width Δy in clock position y . The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $\frac{1}{dE} = \frac{m}{kdk}$ and is given by $\frac{dx(y)m}{k_o}$ here $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the width in x arises because the wave function is spreading as time increases, so that at later y , the wave packet is wider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles with larger mass.

When the back-reaction causes a small disturbance to the particle, the clock records the time-of-arrival. What happens when we wish to make more accurate measurements? Consider the exact transition probability $T = \frac{q}{k} |A_I|^2$, which also determines the probability to stop the clock. The latter is given by

$$T = \sqrt{\frac{E_k + p}{E_k}} \left[\frac{2\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}} \right]^2. \quad (3.60)$$

Since the possible values obtained by p are of the order $1/\Delta_y \equiv 1/\Delta t_A$, the probability to trigger the clock remains of order one only if

$$\bar{E}_k \delta t_A > 1. \quad (3.61)$$

Here δt_A stands for the initial uncertainty in position of the dial \mathbf{y} of the clock, and is interpreted as the accuracy of the clock. \bar{E}_k can be taken as the typical initial kinetic energy of the particle.

In measurements with accuracy better than $1/\bar{E}_k$ the probability to succeed drops to zero like $\sqrt{E_k \delta t_A}$, and the time-of-arrival of most of the particles cannot be detected. Furthermore, the probability distribution of the fraction which has been detected depends on the accuracy δt_A and can become distorted with increased accuracy. This observation becomes apparent in the following simple example. Consider an initial wave packet that is composed of a superposition of two Gaussians centered around $k = k_1$ and $k = k_2 \gg k_1$. Let the classical time-of-arrival of the two Gaussians be t_1 and t_2 respectively. When the inequality (3.61) is satisfied, two peaks around t_1 and t_2 will show up in the final probability distribution. On the other hand, for $\frac{2m}{k_1^2} > \delta t_A > \frac{2m}{k_2^2}$, the time-of-arrival of the less energetic peak will contribute less to the distribution in y , because it is less likely to trigger the clock. Thus, the peak at t_1 will be suppressed. Clearly, when the precision is finer than $1/\bar{E}_k$ we shall obtain a distribution which is considerably different from that obtained for the case $\delta t_A > 1/\bar{E}_k$ when the two peaks contribute equally.

3.3.2 Time-of-Arrival P_∞ for $\delta t_A <$

trigger without including the clock:

$$H_{trigger} = \frac{1}{2m} \mathbf{P}_x^2 + \frac{\alpha}{2} (1 + \sigma_x) \delta(\mathbf{x}). \quad (3.62)$$

The particle interacts with the repulsive Dirac delta function potential at $x = 0$, only if the spin is in the $|\uparrow_x\rangle$ state, or with a vanishing potential if the state is $|\downarrow_x\rangle$. In the limit $\alpha \rightarrow \infty$ the potential becomes totally reflective (Alternatively, one could have considered a barrier of height α^2 and width $1/\alpha$.) In this limit, consider a state of an incoming particle and the trigger in the “on” state: $|\psi\rangle|\uparrow_z\rangle$. This state evolves to

$$|\psi\rangle|\uparrow_z\rangle \rightarrow \frac{1}{\sqrt{2}} \left[|\psi_R\rangle|\uparrow_x\rangle + |\psi_T\rangle|\downarrow_x\rangle \right], \quad (3.63)$$

here ψ_R and ψ_T are the reflected and transmitted wave functions of the particle, respectively.

The latter equation can be rewritten as

$$\frac{1}{2} |\uparrow_z\rangle (|\psi_R\rangle + |\psi_T\rangle) + \frac{1}{2} |\downarrow_z\rangle (|\psi_R\rangle - |\psi_T\rangle) \quad (3.64)$$

Since \uparrow_z denotes the “on” state of the trigger, and \downarrow_z denotes the “off” state, we have flipped the trigger from the “on” state to the “off” state with probability $1/2$ ². Although this model only works half the time, the chance of success does not depend in any way on the system, and in particular, on the particle’s energy. Furthermore, one can construct models where a detector is triggered almost all the time [35], although with some energy dependence in the probability of triggering.

So far we have succeeded in recording the event of arrival to a point. We have no information at all on the time-of-arrival. It is also worth noting that the net energy exchange between the trigger and the particle is zero, i.e., the particle’s energy is unchanged.

The eigenstates of the Hamiltonian in the basis of σ_z are

$$\Psi_L(x) = \begin{pmatrix} e^{ik_\uparrow x} + \phi_{L\uparrow} e^{-ik_\uparrow x} \\ \phi_{L\downarrow} e^{-ik_\downarrow x} \end{pmatrix} e^{ipy}, \quad (3.66)$$

for $x < 0$ and

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} e^{ik_\uparrow x} \\ \phi_{R\downarrow} e^{ik_\downarrow x} \end{pmatrix} e^{ipy}, \quad (3.67)$$

for $x > 0$. Here $k_\uparrow = \sqrt{2m(E - p)} = \sqrt{2mE_k}$ and $k_\downarrow = \sqrt{2mE} = \sqrt{2m(E_k + p)}$.

Matching conditions at $x = 0$ yields

$$\phi_{R\uparrow} = \frac{\frac{2k_\uparrow}{m} - \frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m} - (1 + \frac{k_\uparrow}{k_\downarrow})} \quad (3.68)$$

$$\phi_{R\downarrow} = \frac{k_\uparrow}{k_\downarrow} (\phi_{R\uparrow} - 1) = \frac{\frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m} - (1 + \frac{k_\uparrow}{k_\downarrow})}, \quad (3.69)$$

and

$$\phi_{L\downarrow} = \phi_{R\downarrow} \quad (3.70)$$

$$\phi_{L\uparrow} = \phi_{R\uparrow} - 1. \quad (3.71)$$

We find that in the limit $\alpha \rightarrow \infty$ the transmitted amplitude is

$$\phi_{R\downarrow} = -\phi_{R\uparrow} = \frac{\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}}. \quad (3.72)$$

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3.3.3

There is however one limiting case in which the method does seem to succeed. Consider a narrow wave peaked around k with a width dk . To first order in dk , the probability T_{\downarrow} that the particle is successfully boosted is given by

$$T_{\downarrow} \simeq 1 - \frac{2dk}{k}. \quad (3.80)$$

Therefore in the special case that $\frac{dk}{k} \ll 1$, the transition probability is still close to one. If in this case we know in advance the value of k up to $dk \ll k$, we can indeed use the booster to improve the bound (3.61) on the accuracy.

The reason why this seems to work in this limiting case is as follows. The probability of flipping the particle's spin depends on how long it spends in the magnetic field described by the α term in (3.73). If however, we know beforehand, how long the particle will be in this field, then we can tune the strength of the magnetic field (α) so that the spin gets flipped. The requirement that $dk/k \ll 1$ is thus equivalent to having a small uncertainty in the "interaction time" with this field. In some sense, the measurement is possible, because we know the particle's momentum before hand. Of course, if we have prior knowledge of the particle's momentum, then we could just measure \mathbf{x} and argue that this allows us to calculate the time of arrival through $t_A = mx/p$. We therefore believe that the reason the measurement procedure described above works in this limiting case is because it assumes prior knowledge of the particle's momentum, and we do not believe that one could improve it so that it works for all states. These "booster" measurements cannot be used for general wave functions, and even in the special case above, one still requires some prior information of the incoming wave function.

3.3.4 Gradual triggering of the clock

In order to avoid the reflection found in the previous two models, we shall now replace the sharp step-function interaction between the clock and particle by a more gradual

transition.

When the WKB condition is satisfied

$$\frac{d\lambda(x)}{dx} = \epsilon \ll 1 \quad (3.81)$$

here $\lambda(x)^{-2} = 2m[E_0 - V(x)]$, the reflection amplitude vanishes as

$$\sim \exp(-1/\epsilon^2) \quad (3.82)$$

Solving the equation for the potential with a given ϵ we obtain

$$V_\epsilon(x) = E_0 - \frac{1}{2m\epsilon^2} \frac{1}{x^2} \quad (3.83)$$

Now we observe that any particle with $E \geq E_0$ also satisfies the WKB condition (3.81) above for the

The problem is however that the final value of $t - \mathbf{y}$ does not always correspond to the time-of-arrival since it contains errors due to the affect of the potential $V(x)$ on the

The time-of-arrival can hence be measured provided that $E_k \delta t \gg 1$. On the other hand, when the detector's accuracy is $\delta t < 1/E$, the particle still triggers the clock. However, the measured quantity, A , no longer corresponds to the time-of-arrival. Again, we see that when we ask for too much accuracy, the particle is strongly disturbed and the reading of the clock has nothing to do with the time-of-arrival of a free particle.

3.3.5 General considerations

We have examined several models for a measurement of time-of-arrival and found a limitation,

$$\delta t_A > 1/\bar{E}_k, \quad (3.91)$$

on the accuracy that t_A can be measured. Is this limitation a general feature of quantum mechanics?

First we should notice that eq. (3.91) does not seem to follow from the uncertainty principle. Unlike the uncertainty principle, whose origin is kinematic, (3.91) follows from the nature of the *dynamic* evolution of the system during a measurement. Furthermore, here we are considering a restriction on the accuracy (not uncertainty) of a single measurement. While it is difficult to provide a general proof, in the following we shall indicate why (3.91) is expected to hold under more general circumstances.

Let us examine the basic features that gave rise to (3.91). In the toy models considered in Sections 3.3.1 and 3.3.2, the clock and the particle had to exchange energy $p_y \sim 1/\delta t_A$. As a result, the effective interaction by which the clock switches off, looks from the point of view of the particle like a step function potential. This led to “non-detection” when (3.91) was violated.

Can we avoid this energy exchange between the particle and the clock? Let us try to deliver this energy to some other system without modifying the energy of the particle.

For example consider the following Hamiltonian for a clock with a reservoir:

$$H = \frac{\mathbf{P}_x^2}{2m} + \theta(-\mathbf{x})H_c + H_{res} + V_{res}\theta(\mathbf{x}) \quad (3.92)$$

The idea is that when the clock stops, it dumps its energy into the reservoir, which may include many other degrees of freedom, instead of delivering it to the particle. In this model, the particle is coupled directly to the clock and reservoir, however we could as well use the idea of Section 3.3.2 above. In this case:

$$H = \frac{\mathbf{P}_x^2}{2m} + \frac{\alpha}{2}(1 + \sigma_x)\delta(\mathbf{x}) + \frac{1}{2}(1 + \sigma_z)H_c + H_{res} + \frac{1}{2}(1 - \sigma_z)V_{res}. \quad (3.93)$$

The particle detector has the role of providing a coupling between the clock and reservoir.

Note we notice that in order to transfer the clock's energy to the reservoir without affecting the free particle, we must also prepare the clock and reservoir in an initial state that satisfies the condition

$$H_c - V_{res} = 0 \quad (3.94)$$

However this condition does not commute with the clock time variable \mathbf{y} . We can measure initially $\mathbf{y} - \mathbf{R}$, where R is a collective degree of freedom of the reservoir such that $[\mathbf{R}, V_{res}] = i$, but in this case we shall not gain information on the time-of-arrival y since R is unknown. We therefore see that in the case of a sharp transition, i.e. for a localized interaction with the particle, one cannot avoid a shift in the particle's energy. The “non-triggering” (or reflection) effect cannot be avoided.

We have also seen that the idea of boosting the particle “just before” it reaches the detector, fails in the general case. What happens in this case is that while the detection rate increase, one generally destroys the initial information stored in the incoming wave packet. Thus although higher accuracy measurements are not possible, they do not reflect directly the time-of-arrival of the initial wave packet.

